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USING NUMBER SENSE TO COMPAR FRACTI

Third-grade teachers found that giving particular attention to the use of real-world contexts, mental imagery, and manipulatives brought success to problem solving as students moved from using models to reasoning.

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By Wendy S. Bray and Laura Abreu-Sanchez

One mathematical focus for third graders is to develop deep understanding of fractions and fraction equivalence, including comparing fractions through use of models and reasoning strategies (NCTM 2006). Before reading further, consider how you might solve the following problem: Which fraction is greater, 14/24 or 17/36?

The initial impulse of many adults is to solve fraction comparison problems by finding common denominators or performing cross multiplication. Yet the fractions in the question above make those methods cumbersome. Drawing pictures is another possibility, but creating drawings useful for comparison would be difficult without computer models. Which other strategies might we use to compare 14/24 and 17/36? Consider the following strategy:

> Although both fractions are close to 1/2, 14/24 is more than 1/2 because 12/24 is exactly 1/2 and 14/24 is greater than that. Likewise, 17/36 is less than 1/2 because 18/36 is exactly 1/2 and 17/36 is less than that. So 14/24 is greater than 17/36.

This strategy of comparing to one-half not only is more efficient than finding common denominators or using cross multiplication for the problem posed, but also requires conceptual and relational ways of thinking about fractions.

Although van de Walle's four conceptually based reasoning strategies for comparing fractions (see **sidebar** on p. 92) will not work for every possible fraction comparison problem, the strategies provide a powerful framework to support students' initial development of strategies for comparing

Fraction circles offer a context to discuss fraction names and review fraction symbols. Rather than saying, "Two-fourths equals one-half," students might say, "Two blues are the same as one yellow."

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Reasoning strategies for comparing fractions

Elementary and Middle School Mathematics: Teaching Developmentally (van de Walle 2007) identifies four conceptually based reasoning strategies for comparing fractions:

1. **More same-sized parts** (same denominators)
   To compare 3/10 and 5/10, recognize that both fractions involve tenth-sized parts. Since the size of the parts is the same for both fractions, comparison can focus on the number of parts. In this case, five parts is more than three parts. So 5/10 is the greater fraction.
2. **Same number but different-sized parts** (same numerators)
   To compare 3/8 and 3/10, notice that both fractions involve three parts, but the size of the parts differ. Since eighth-sized parts are larger than tenth-sized parts, 3/8 is greater than 3/10.
3. **More and less than one-half or one whole** (comparison to benchmarks)
   To compare 7/8 and 8/7, consider these fractions in relationship to the benchmark of one whole unit. Since 7/8 is less than 1 and 8/7 is more than 1, then 8/7 is greater than 7/8.
4. **Closeness to one-half or one whole** (distance from benchmarks)
   To compare 5/6 and 7/8, notice that both fractions are one part away from one whole. Focus on the comparative size of that missing part. Since the missing 1/8 part is smaller than the missing 1/6 part, 7/8 is closer to 1 and is therefore the greater fraction. Similarly, 6/10 is a greater fraction than 7/12, because both these fractions have one part more than one-half. That is, 6/10 is 1/2 + 1/10, and 7/12 is 1/2 + 1/12; so 1/10 is greater than 1/12.

fractions in ways that simultaneously emphasize fraction number sense. Authors Wendy S. Bray and Laura Abreu-Sanchez furnish an overview of these conceptually based reasoning strategies followed by a description and a discussion of their efforts—and what they learned along the way—to support third graders' use of their growing conceptual knowledge of fractions to reason through fraction comparison problems, with and without the support of models.

Setting the stage for comparing fractions

Before focusing on comparing and ordering fractions, students in Abreu-Sanchez's third-grade classroom spent two weeks exploring how to name and represent fractional parts through equal-sharing and part-whole problems involving area, linear, and set models for fractions. (See van de Walle [2007] for information on using such problems.) Students regularly worked with peers to discuss and justify mathematical solutions, a process that deepened their understanding of part-whole fraction concepts and helped prepare them to compare fractions.

As Abreu-Sanchez and Bray shifted to an instructional focus on comparing fractions, their primary goal was to promote students' development of reasoning strategies to compare fractions by encouraging students to construct mental images built on experiences with fraction manipulatives. To best accomplish this goal, the teachers spent extended time with the circle model before exploring other fractional models. The arc of a circle model's pie-wedge-shaped fraction pieces helps students "see" the relationship between a fractional part and the one-half and one-whole benchmarks because circle pieces—unlike other manipulatives—offer only one way to build a given fraction (see **fig. 1**).

The next step was to develop a context for exploring fraction comparison using the circle model manipulatives. Bray and Abreu-Sanchez developed a series of real-world problems that incorporated students' names and were designed to evoke discussion of the reasoning strategies of comparing the same numerator or the same denominator and using the benchmark of one-half.

In the classroom

On the first day of instruction, fraction circle manipulative kits were distributed to table groups, and students were directed to explore the kits and keep track of their mathematical observations. The kits included plastic circles of different colors divided into halves, thirds, fourths, fifths, sixths, eighths, tenths, and twelfths as well as a piece representing one whole. Students soon began to find equivalent fractions. But rather than identifying 2/4 as equivalent to 1/2, students might say, “Two blues are the same as one yellow.” This offered a context for discussing how to use fraction names to describe the different colored pieces and to review the meaning of fraction symbols.

To elicit a discussion of the same-numerator strategy, students next received a problem in which the numerator was a unit fraction:

> At a special pizza luncheon, Eduardo and David both ate one slice of pizza. Eduardo ate one slice from a large pepperoni pizza that was cut into six equal pieces. David ate one slice from a large cheese pizza that was cut into ten equal pieces.
>
> - **What fraction** of a large pizza did each boy get?
> - **Which** boy got more pizza? How do you know?

The teachers encouraged students to first attempt to picture the problem and solution in their heads and then to work with the fraction circles and their tablemates to determine or verify solutions. After table groups had time to exchange ideas, students discussed their predictions and findings as a class.

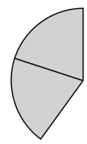
Teacher: So before looking at the fraction circles, some of you thought that one-tenth of a pizza would be greater and others thought that one-sixth of a pizza would be greater. What did you find out when you looked at the fraction circles?

Class: One-sixth [is the greater fraction].

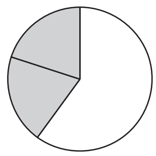
Teacher: Can somebody who predicted that one-tenth would be the greater fraction talk about why you thought that? Thanks, Sheena.

FIGURE 1

The circle model and the bar model suggest these mental images.

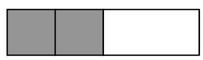


This figure only suggests $\frac{2}{5}$

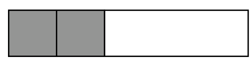


and makes clear that $\frac{2}{5}$ is close to but less than $\frac{1}{2}$.

This figure could suggest $\frac{2}{4}$ or



$\frac{2}{5}$ or



other fractions, depending on the whole.

The relationship of the fractional part to $\frac{1}{2}$ and 1 is not clear without additional pieces.

Sheena: Well, because ten is a bigger number than six.

Teacher: What did your table [group members] find out when you looked at the fraction circles?

Sheena: That one-sixth is bigger than one-tenth.

Teacher: Why do you think that one-sixth of a pizza turned out to be larger than one-tenth? Why is it that tenth-sized pieces are smaller than sixth-sized pieces?

Sheena: Well, you have to share the pizza with ten people. So each person gets less. If there are less people sharing, you get more pizza.

Teacher: Can someone else explain—in your own words—what Sheena is saying? John?

John:	It is like opposites. When the bottom number [of a fraction] is bigger, it means the pieces are smaller.
Teacher:	OK. Why is that the case? Someone else. Arlene?
Arlene:	Because you have to split it up for more people.

This discussion highlighted a common misconception about fractions and led students to justify correct ideas that are the foundation for the common-numerator reasoning strategy.

Next, the teachers posed follow-up problems, such as those below, designed to encourage application of the ideas discussed in relation to the initial problem:

- **Who** would have eaten more if Eduardo ate 1/5 of a large pizza and David ate 1/4 of one?
- **Which** is more, 1/10 or 1/12?

With each follow-up problem, students were encouraged to picture the fractions mentally and reason about their relative size before manipulating the fraction circles to find or verify solutions. Whole-class discussions stressed the importance of considering both the number of pieces (the numerator) and the size of pieces (the denominator). The teachers emphasized that students could focus on the size of the pieces in these problems because the number of pieces was the same. Finally, a discussion of additional follow-up problems took place, involving fractions that could not be represented with the fraction circles (e.g., Which is more, 1/20 or 1/18?).

In the remainder of the first lesson, students explored problems that follow the instructional sequence of conjecture-verify-discuss:

- **Conjecture:** Use mental imagery.
- **Verify:** Use fraction circles to show your solutions and discuss them with your tablemates.
- **Discuss:** Participate in the whole-class discussion of solutions, and be ready to justify your ideas.

The Christina problem in **table 1** extended the same-numerator strategy to fractions with numerators greater than one. The Devyn problem provided a context to explore the same-denominator strategy while also heading off a common overgeneralization. Several students initially conjectured incorrectly that 2/10 would be more than 3/10, reasoning that the smaller number (i.e., the numerator 2 in 2/10) was going to have larger pieces than the fraction with a 3.

As instruction continued, the teachers sought to address the mistaken notion (of smaller numerators equating to larger pieces) by intentionally juxtaposing problems with same numerators with those having same denominators in order to push students to mentally coordinate conceptual understanding of numerators

TABLE 1

Additional problems evoke same-numerator and same-denominator strategies.

Strategy	Initial Problem	Follow-up Problems
Same Numerator (nonunit fractions)	Christina ate three slices of a large pepperoni pizza that was cut in six equal pieces. Maria ate three slices of a large mushroom pizza that was cut in eight equal pieces. • What fraction of a large pizza did each girl eat? • Which girl ate more? How do you know?	Which is more? $\frac{2}{5}$ or $\frac{2}{3}$ $\frac{7}{10}$ or $\frac{7}{12}$ $\frac{18}{20}$ or $\frac{18}{30}$
Same Denominator	Devyn ate $\frac{2}{10}$ of a large cheese pizza, and Hannah ate $\frac{3}{10}$ of a large cheese pizza. • Who ate more cheese pizza? • How do you know?	Which is more? $\frac{6}{8}$ or $\frac{5}{8}$ $\frac{2}{3}$ or $\frac{1}{3}$ $\frac{16}{20}$ or $\frac{18}{20}$

TABLE 2

Some problems evoke comparison to the benchmark $\frac{1}{2}$.

Problem Type	Initial Problem	Follow-up Problems
Compare to $\frac{1}{2}$	Sophia ate $\frac{4}{10}$ of a large cheese pizza. Erick ate $\frac{1}{2}$ of a large mushroom pizza. • Who ate more pizza? • How do you know?	Which is more? $\frac{1}{2}$ or $\frac{5}{8}$ $\frac{1}{2}$ or $\frac{5}{12}$ $\frac{1}{2}$ or $\frac{9}{20}$
Compare to a fraction that is equivalent to $\frac{1}{2}$	Alan ate $\frac{4}{8}$ of a pepperoni pizza, and Jason ate $\frac{5}{12}$ of a cheese pizza. • Who ate more pizza? • How do you know?	Which is more? $\frac{5}{8}$ or $\frac{2}{4}$ $\frac{3}{6}$ or $\frac{4}{12}$ $\frac{23}{50}$ or $\frac{10}{20}$
Compare fractions on either side of $\frac{1}{2}$	Sarah ate $\frac{4}{10}$ of a mushroom pizza, and Carson ate $\frac{3}{4}$ of a sausage pizza. • Who ate more pizza? • How do you know?	Which is more? $\frac{3}{8}$ or $\frac{4}{6}$ $\frac{7}{12}$ or $\frac{2}{5}$ $\frac{15}{20}$ or $\frac{18}{40}$

and denominators when they reason about the relative size of fractions.

After some additional follow-up and checkpoint assessment of these first two strategies, activities designed to emphasize benchmark fractions were introduced. First, students manipulated fraction circles to generate a list of fractions equivalent to one-half. By exploring the pattern among these fractions, students were able to form the generalization that fractions are equivalent to one-half when the numerator is half the denominator. Students also noticed that only fractions with even denominators can be equivalent to one-half. Students discussed how two-fifths is half a fifth less than one-half and that three-fifths is half of a fifth more than one-half. They used their discoveries to suggest other fractions equivalent to one-half that could not be represented by the fraction kit (e.g., 25/50).

FIGURE 2

These sets of fractions can be used to explore the benchmarks of $0, \frac{1}{2}$, and 1.

Set A: $\frac{9}{10}, \frac{1}{12}, \frac{5}{8}, \frac{2}{10}, \frac{5}{12}, \frac{5}{6}$

Set B: $\frac{2}{12}, \frac{4}{4}, \frac{4}{10}, \frac{3}{6}, \frac{1}{8}, \frac{10}{12}$

Then students received two sets of written fractions (see **fig. 2**) and worked in teams to determine whether each fraction was closest to zero, to one-half, or to one whole. As before, students were directed to first conjecture about solutions mentally and then to talk about their reasoning with tablemates and investigate with fraction circles. In discussion, the teachers pressed students to be explicit about fraction equivalencies and relationships to benchmark fractions. For the fraction 9/10, students were prompted to justify that because 5/10 is equal to 1/2 and 10/10 is equal to one whole, 9/10 is 4/10 more than 1/2 and 1/10 away from one whole.

After this introduction to benchmarks, students used the established process of conjecture-verify-discuss to work through a series of increasingly sophisticated problems designed to evoke the use of comparing fractions to the benchmark of one-half (see the problems in **table 2**). Students readily

discussed their reasoning for the benchmark-of-one-half strategy for problems in which one of the fractions was 1/2 (e.g., the Sophia problem) or equivalent to 1/2 (e.g., the Alan problem). For example, Hannah explained that Alan's 4/8 of a pizza is more than Jason's 5/12 of a pizza because 4/8 is the same as 1/2 and 5/12 is less than 1/2. Prompted to justify how she knew that 4/8 is the same as 1/2, she explained that four is half of eight. Then she explained that 5/12 is less than 1/2 because six is half of twelve and 5/12 is one slice less than 6/12.

Despite ease with initial benchmark problems, only a few students approached problems involving comparison of fractions on either side of one-half (e.g., the Sarah problem) by making comparisons to one-half. Several students responded to the Sarah problem by explaining that 3/4 of a pizza is greater than 4/10 because 3/4 is close to a whole and 4/10 is closer to zero. This is an accurate and justifiable line of reasoning, given the distance between these fractions and their relationship to the benchmarks of zero and one. Other students reasoned that 3/4 is greater because it is one piece from a whole and 4/10 is many pieces from a whole. Although this line of reasoning yielded a correct answer for the Sarah problem, it would not be reliable in all comparison problems involving fractions with different denominators (e.g., 5/8 is greater than 6/10). When students used this strategy, the teachers told them that the strategy would not work for all fraction-comparison problems and challenged them to verify their answer with a different reasoning strategy.

Students learned that creating visual models to concretely compare fractions is cumbersome but that reasoning using a benchmark of one-half is efficient.

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In the days that followed, students had further opportunities to use reasoning strategies for comparing and ordering fractions using additional contextualized problems and a fraction comparison card game similar to War. In some of these activities, students were exposed to comparing fractions with models other than circles, including fraction bars and the number line.

What the teachers learned

Assessing the ability to compare and order fractions, Bray and Abreu-Sanchez found that most students were able to use the reasoning strategies on which they had focused to orally justify their solutions. However, drawings were the most frequent strategy recorded as explanation on written assessments. At first the teachers were disappointed by this outcome but later came to appreciate the ways that students were using reasoning to create and think about their drawings. For example, when drawing 4/10, students' drawings suggest that they first divided a whole into half and then attempted to make five equal parts from each half. This drawing strategy suggests an understanding of the relationship between 1/2, 5/10, and 4/10. When asked to compare the fractions 1/10 and 1/12, students' drawings were not accurate enough to be useful. Nonetheless, students consistently identified 1/10 as the greater fraction, suggesting their ability to reason about the relative size of these fractions. Whereas these findings are generally encouraging, Bray and Abreu-Sanchez plan to give greater attention to eliciting written explanations of reasoning with future students.

As anticipated, the authors found that the fraction circle manipulatives furnish a useful foundation for developing mental imagery for fraction comparison. Yet they also concluded that some students might have benefited from more work with the fraction circles at the concrete level before the emphasis on mental imagery. Next year the teachers will add a day or two

at the beginning of the lesson sequence in which students engage in activities focused on gaining greater fluency and comfort with fraction circle manipulatives. They also intend to add more opportunities throughout the lesson sequence for students to draw connections between the circle model and other fraction models, particularly the number line.

Finally, they intend to add activities at the end of the sequence that prompt discussion of the distance-from-a-benchmark strategy. Posing comparison problems with fractions that are close together on the same side of a benchmark would force students to consider more precisely the relationship between a benchmark and the fractions being compared. For example, if asked to compare 6/10 and 5/8, most students would likely notice that both fractions are a little more than one-half and do not have common numerators or denominators. To determine the greater fraction, students would need to figure out that 6/10 is 1/10 more than one-half and that 5/8 is 1/8 more than one-half. Then they could reason that 5/8 is the greater fraction because 1/8 is more than 1/10. Likewise, to compare 5/6 and 7/8, students would need to identify that both fractions are one "piece" away from one whole and that the 1/8-sized piece missing from 7/8 places it slightly closer to a whole than the 1/6-sized piece that is missing from 5/6.

The distance-from-a-benchmark strategy is clearly more difficult than the others explored in the lesson sequence, but time spent exploring this strategy will help to deepen students' understanding of and coordination among the other strategies as well as support their future development of invented strategies for adding and subtracting fractions.

Final thoughts

Traditionally, students have been taught to compare fractions by first creating visual models and then by converting to common denominators. Although children can find answers to fraction comparison problems using models, these experiences alone do not prepare students to compare fractions without models. Children have rarely understood the strategy of converting to common denominators, which is cumbersome with uncommon fractions. In contrast, the authors found that emphasizing comparing fractions through use of mental imagery and reasoning strategies is highly supportive of developing students' ability to compare fractions while simultaneously increasing their fraction number sense. When these teachers encouraged the use of reasoning strategies, students appeared to stretch their thinking to mentally coordinate conceptual information about numerators and denominators to a greater extent than students had in previous years, when they had primarily used pictures to compare fractions. Such experiences with reasoning about fractions through use of mental imagery and fraction relationships will serve as a solid foundation to support these students' continuing development of deep rational number understanding in the grades to come.

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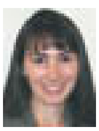

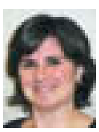

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Reflect and discuss

"Using number sense to compare fractions"

Reflective teaching is a process of self-observation and self-evaluation. Look at your classroom practice, think about what you do and why you do it, and then evaluate whether it works. The following questions and prompts related to this article by Wendy S. Bray and Laura Abreu-Sanchez are to aid you in reflecting independently on the article, discussing it with your colleagues, and considering how the authors' ideas might benefit your own classroom practice.

1. Which types of manipulatives have you found to be effective when encouraging students to think about fractions? What do you think about the authors' suggestion to use circle models?
2. How does encouraging students to think about fraction comparisons develop reasoning that builds foundations for future work with fractions?
3. These authors suggest carefully sequencing instruction to expose students to one fraction-comparison strategy at a time. Compare and contrast this approach to one that encourages students to use a variety of strategies at once and that focuses discussion on the benefits of each strategy and the connections among them.

Tell us how you used "Reflect and discuss" as part of your professional development. Submit letters to *Teaching Children Mathematics* at **tcm@nctm.org**. Include "readers exchange" in the subject line. Find more information at **tcm.msubmit.net**.